Arkhipov-Bezrukavnikov (finished)

Fully faithfulness of Av_ψ

Define $Av_I = \mathrm{Ind}^{I}_{I\cap I\bar{u}} : D_{IW} \to D_I$. Lemma ${}^pH^{\ell(w_0)}(Av_I(\Delta_0)) = j_{w_0!}$

From the definition, $Av_I(\Delta_0)$ is supported in finite flag.

$I\cap I\bar{u}$ is unipotent, hence get the adjointness.

$$\mathrm{Hom}^{\bullet}_{D_I}(X, Av_I(\Delta_0)) = \mathrm{Hom}^{\bullet}_{D}(\mathrm{oblv}(X), \Delta_0)$$

Recall $\pi_{\alpha *}\Delta_0 = 0$, we know $\mathrm{Hom}^{\bullet}_D(L_w, \Delta_0) = 0$, hence

$$\mathrm{Hom}^{\bullet}_{D_I}(L_w, Av_I(\Delta_0)[\ell(w_0)]) = 0 \quad \text{for all } w \neq e,\ w \in W_f.$$

It is clear that $\mathrm{Hom}^{\bullet}_D(L_e, \Delta_0[\ell(w_0)]) = H^{\bullet}(j_e^! \Delta_0[\ell(w_0)]) = \Lambda$.

Hence $\mathrm{Hom}^{\bullet}_{D_I}(j_{w!}, Av_I(\Delta_0)[\ell(w_0)]) = \Lambda$.

Furthermore, there's the map $j_{w_0!} \to Av_I(\Delta_0)[\ell(w_0)]$ such that

$\mathrm{Hom}_{D_I}(j_{w!}, j_{w_0!}) \to \mathrm{Hom}_{D_I}(j_{w!}, Av_I(\Delta_0)[\ell(w_0)])$ is an isomorphism.

Furthermore, $\mathrm{Hom}_{D_I}(j_{w!}, j_{w_0!}[1]) = 0$ implies

$$\mathrm{cone}(j_{w_0!} \to Av_I(\Delta_0)[\ell(w_0)]) \in {}^pD^{>0}. \qquad \square$$

Here strongly use the fact that we are in I-equivariant category.

Define $F'(-) = pr_f \circ {}^pH^0(Av_I(-)[\ell(w_0)])$. Lemma $F' \circ Av_\psi = id$.

Note that $\Delta_0 * L_w * \mathcal{F} = 0$, $w \neq e$, we have

$L_w * \mathcal{F} \in \ker Av_\psi$, hence $pr_f(L_w * \mathcal{F}) = 0$, $w \neq e$ and $= pr_f(\mathcal{F})$, $w = e$.

Thus convolution descends to ${}^fP_I^0 \times {}^fP_I \to {}^fP_I$.

For $\mathcal{F} \in P_I$, we have ${}^pH^0(Av_I(\Delta_0)[\ell(w_0)] * \mathcal{F}) = {}^pH^0(j_{w_0!} * \mathcal{F}) = \mathcal{F}$.

Thus $F' \circ Av_\psi(pr_f \mathcal{F}) = pr_f(\mathcal{F})$. $\square$

Together with the fact that Av_ψ send irreducible objects to non-isomorphic irreducible objects shows that it is full embedding.

Back to Last time

Lemma if $\mathrm{Stalk}(\mathrm{Fiw}(V_1))$ and $\mathrm{Stalk}(\mathrm{Fiw}(V_2))$ are in degree zero, then $\mathrm{Stalk}(\mathrm{Fiw}(V_1 \otimes V_2))$ is also.

$\Delta_0 * Z(V_1)$ has standard filtration, hence suffices to show $\Delta_\mu * Z(V_2)$ is in degree zero. This one is perverse, hence stalk is in degree ≤ 0. Note $\Delta_\mu * Z(V_2) = \Delta_0 * Z(V_2) * j_{\mu !}$ has a filtration of $\Delta_\lambda * j_{\mu !} = \Delta_0 * (j_{\lambda !} * j_{\mu !})$, only need to show $j_{\lambda !} * j_{\mu !} \in \langle j_{w!}[i],\ w \in W,\ i \leq 0 \rangle$, which is proved by induction.

Lemma For $w \in W_f$, $V \in \mathrm{Rep}(G^\vee)$, we have $\mathrm{stalk}_\lambda(\mathrm{Fiw}(V)) = \mathrm{stalk}_{w(\lambda)}(\mathrm{Fiw}(V))$.

It suffices to show the case for $w = s$ is simple reflection and $s(\lambda) \leq \lambda$.

For $X \in D_{Iw}$, the exact sequence $0 \to L_s \to j_{s*} \to L_e \to 0$ gives the triangle $\mathrm{stalk}_\lambda(X * L_s) \to \mathrm{stalk}_\lambda(X * j_{s*}) \to \mathrm{stalk}_\lambda(X) \xrightarrow{+1}$.

Recall $\mathrm{stalk}_\lambda(X * j_{s*}) = \mathrm{Hom}^\bullet(X * j_{s*}, \nabla_\lambda) = \mathrm{Hom}^\bullet(X, \nabla_\lambda * j_{s!})$

Then $\nabla_\lambda * j_{s!} = \Delta_0 * j_{\lambda *} * j_{s!} = \Delta_0 * j_{\lambda s *} = \nabla_{s(\lambda)}$ implies the triangle $\mathrm{stalk}_\lambda(X * L_s) \to \mathrm{stalk}_{s(\lambda)}(X) \to \mathrm{stalk}_\lambda(X) \xrightarrow{+1}$

When $X = \mathrm{Fiw}(V)$, $X * L_s = \Delta_0 * Z(V) * L_s = \Delta_0 * L_s * Z(V) = 0$,

Hence $\mathrm{stalk}_{s(\lambda)}(X) = \mathrm{stalk}_\lambda(X)$ in this case.

Cor if λ is minuscule, then $\mathrm{Stalk}(\mathrm{Fiw}(V_\lambda))$ is in deg 0.

From the construction of Z_λ, $\dim \mathrm{supp}\, Z_\lambda = \dim \mathrm{Gr}_\lambda = \dim \mathrm{Fl}_\lambda$.

$\mathrm{Fl}_\lambda \subset \mathrm{supp}\, Z_\lambda$ shows that Fl_λ is open in $\mathrm{supp}\, Z_\lambda$.

Hence the stalk of Z_λ at λ is Λ since $IC(\mathrm{Gr}_\lambda)|_{\mathrm{Gr}_\lambda}$ is so.

Taking $\Delta_0 * -$, we know Fl^λ is open in $\mathrm{supp}\, \mathrm{Fiw}(V_\lambda)$, and $\mathrm{stalk}_\lambda(\mathrm{Fiw}(V_\lambda))$ is in deg 0 and has dim one.

The lemma shows that $\mathrm{stalk}_{w(\lambda)}$ has the same property.

λ is minuscule $\Rightarrow$ the only non-zero stalks are these $w(\lambda)$, $w \in W_f$.

Stalk $(\mathrm{Fw}(V_\lambda))$ is concentrated in degree 0 for λ quasi-minisucle.

For the same reason, it suffices to consider the stalk at 0.

The exact triangle with respect to $\overline{\mathcal{Fl}}^\circ \hookrightarrow \mathcal{Fl}$ shows that

$\mathrm{Stalk}_0(\mathrm{Fw}(V_\lambda))$ is concentrated in degree $[-1,0]$.

Recall $\sum(-1)^i \dim H^i(\mathrm{Stalk}_0(\mathrm{Fw}(V_\lambda))) = [0:V_\lambda] = \dim V_\lambda^h = \dim V_\lambda^{N_0}$.

for the regular sl_2 triple (h, N_0, f).

we have $H^0(\mathrm{stalk}_0(\mathrm{Fw}(V_\lambda))) = \mathrm{Hom}(\mathrm{Fw}(V_\lambda), \Delta_0)$

$= \mathrm{Hom}_{fP_1}(Z_\lambda, L_e) = \mathrm{Hom}_{fP_1}((Z_\lambda)_m, L_e) \hookrightarrow \mathrm{Hom}_{P_1^\circ}((Z_\lambda)_m, L_e)$

where m is the extension of the monodromy such that

$m_L = 0$ for irreducible $L \in P_1$.

For the inclusion $\mathrm{Rep}(Z_{G^\vee}(N_0)) \hookrightarrow P_1^\circ$ send N_0 to m,

we know $(Z_\lambda)_m$ has exactly length $\dim V_\lambda^{N_0}$ in P_1°.

In conclusion, $\dim H^0(\mathrm{stalk}_0(\mathrm{Fw}(V_\lambda))) \le \sum(-1)^i \dim H^i(\mathrm{Stalk}_0(\mathrm{Fw}(V_\lambda)))$

implies the stalk is concentrated in degree 0.